Area and Definite Integral

Average speed \( f(x) = 1 \)

Area and distance

\[ S = \int_0^1 f(x) \, dx = f(0)(1-0) = 1 \]

\( f(x) = 1 - 4(x-\frac{1}{2})^2 \)

Changed speed

Area and distance

\[ S_{\text{exact}} = \int_0^1 f(x) \, dx \]

\[ S_{\text{approx}} = \sum_{i=0}^{6} f(x_i) \Delta x + \sum_{i=4}^{6} f(x_i) \Delta x \]

\[ \Delta x = x_6 - x_0 = \frac{1-0}{6} \]
\[ S_{\text{out}} = \sum_{i=1}^{3} f(x_i) \Delta x + \sum_{i=3}^{5} f(x_i) \Delta x \]

\[ \Delta x = x_i - x_{i-1} = \frac{1}{6} \]

\[ S_{\text{mix}} = \sum_{i=0}^{5} f(x_i) \Delta x \]

\[ \Delta x = x_i - x_{i-1} = \frac{1}{6} \]

We can observe that

\[ S_{\text{in}} \leq \cdots \leq S_{\text{in}}^b \leq \cdots \leq S_{\text{exact}} \leq \cdots \leq S_{\text{out}}^b \leq \cdots \leq S_{\text{out}} \]

\[ \Rightarrow S_{\text{exact}} = \lim_{n \to \infty} S_{\text{in}}^n = \lim_{n \to \infty} S_{\text{out}}^n \]

Because \( S_{\text{in}} \leq S_{\text{mix}} \leq S_{\text{out}} \),

\[ S_{\text{exact}} = \lim_{n \to \infty} S_{\text{mix}}^n \]
If \( f \) is defined on the interval \([a, b]\), the definite integral of \( f \) from \( a \) to \( b \) is given by

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x
\]

provided the limit exists, where \( \Delta x = \frac{b-a}{n} \) and \( x_i \) is any value of \( x \) in the \( i \)th interval, i.e., \( x_{i-1} \leq x_i \leq x_i \), \( x_i = x_{i-1} + \Delta x \), \( x_0 = a \)

**Total Change in \( F(x) \) (Here \( F'(x) = f(x) \))**

If \( f(x) \) gives the rate of change of \( F(x) \) for \( x \) in \([a, b]\), then the total change in \( F(x) \) as \( x \) goes from \( a \) to \( b \) is given by

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x = \int_a^b f(x) \, dx
\]

**Example:**

\( f(x) = e^x \)

\[
\int_1^e e^x \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x
\]

\( x_i = \frac{i-1}{n} \)